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<u>Real</u> <u>Analysis</u>: De Morgan's and Bertrand Test.

Q.1. State and prove De Morgan's and Bertrand's Test.

Soln. Statement:→ The positive term series Σu_n Converges or diverges according as

$$\lim_{n\to\infty}\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right]>1 \text{ or } <1.$$

Proof: We compare the given series Σu_n with the auxiliary series Σv_n, where $v_n=\frac{1}{n(\log n)^p}$.

We know that Σv_n is Convergent or divergent according as $p>$ or <1.

Hence, by comparison test, the given series Σu_n is convergent or divergent according as

$$\frac{u_n}{u_{n+1}} > \text{ or } < \frac{v_n}{v_{n+1}}$$

or, $$\frac{u_n}{u_{n+1}} > \text{ or } < \frac{\frac{1}{n(\log n)^p}}{\frac{1}{(n+1)\{\log(n+1)\}^p}}$$

or, $$\frac{u_n}{u_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)\left[\frac{\log\{n(1+\frac{1}{n})\}}{\log n}\right]^p$$

or, $$\frac{u_n}{u_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)\left\{\frac{\log n+\log(1+\frac{1}{n})}{\log n}\right\}^p$$

or, $$\frac{u_n}{u_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)\left\{\frac{\log n+\frac{1}{n}-\frac{1}{2n^2}+\cdots \text{ to } \infty}{\log n}\right\}^p$$

or, $$\frac{u_n}{u_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)\left\{1+\frac{1}{n\log n}-\frac{1}{2n^2\log n}+\cdots \text{ to } \infty\right\}^p$$

or, $$\frac{u_n}{u_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)\left(1+\frac{p}{n\log n}-\cdots \text{ to } \infty\right)$$
[By Binomial Theorem]

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or, $\frac{u_n}{u_{n+1}} > $ or $< 1+\frac{1}{n}+\frac{p}{n\log n}+\frac{p}{n^2\log n}+\cdots$ to ∞.

or, $\frac{u_n}{u_{n+1}}-1 > $ or $< \frac{1}{n}\left\{1+\frac{p}{\log n}+\frac{p}{n\log n}+\cdots \text{ to } \infty\right\}$

or, $n\left(\frac{u_n}{u_{n+1}}-1\right) > $ or $< 1+\frac{p}{\log n}+\frac{p}{n\log n}+\cdots$ to ∞

or, $n\left(\frac{u_n}{u_{n+1}}-1\right)-1 > $ or $< p\left(\frac{1}{\log n}+\frac{1}{n\log n}+\cdots \text{ to } \infty\right)$

or, $\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n > $ or $< p\left(1+\frac{1}{n}+\cdots \text{ to } \infty\right)$

proceeding to limits as $n\to\infty$, we get

$$\lim_{n\to\infty}\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right] > \text{ or } < p$$

or, $\lim_{n\to\infty}\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right] > $ or < 1.

Hence, the test, proved.

Q.2. Examine the convergence for $x>0$:

$$\frac{a}{b}+\frac{a(a+1)}{b(b+1)}x+\cdots+\frac{a(a+1)(a+2)\cdots(a+n)}{b(b+1)(b+2)\cdots(b+n)}x^n+\cdots \text{ to } \infty.$$

Soln. Let the general term of the given series be denoted by u_n.

Then, $u_n=\frac{a(a+1)(a+2)\cdots(a+n-1)}{b(b+1)(b+2)\cdots(b+n-1)}x^n$

and $u_{n+1}=\frac{a(a+1)(a+2)\cdots(a+n-1)(a+n)}{b(b+1)(b+2)\cdots(b+n-1)(b+n)}x^{n+1}$

$\therefore \frac{u_n}{u_{n+1}}=\frac{b+n}{a+n}\frac{1}{x}=\frac{\frac{b}{n}+1}{\frac{a}{n}+1}\frac{1}{x}$, $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\frac{1}{x}$.

Hence, by D'Alembert's ratio test, the given series is convergent or divergent according as $\frac{1}{x}>1$ or $\frac{1}{x}<1$, that is, $x<1$ or $x>1$.

When $x=1$, then this test fails.

Let $a\geq b$, then $\frac{u_n}{u_{n+1}}\leq 1$ or $u_n\leq u_{n+1}$ for all n.

$\therefore u_1+u_2+\cdots+u_n \geq nu_1$, $\lim_{n\to\infty}(u_1+u_2+\cdots+u_n)\to\infty$.

$\therefore \sum u_n$ is divergent.

Let $a<b$ and $x=1$. Now we apply Raabe's test.

$$\frac{u_n}{u_{n+1}}-1=\frac{b+n}{a+n}-1=\frac{b-a}{a+n} \quad \text{or, } n\left(\frac{u_n}{u_{n+1}}-1\right)=\frac{bn-an}{a+n}=\frac{b-a}{\frac{a}{n}+1}$$

or, $\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}=b-a$. Hence, by Raabe's test, the given series is convergent or divergent according as $b-a>1$ or <1.

Raabe's test fails when $b-a=1$.

Now, we apply De Morgan and Bertrand's test.

$$\text{Now, } n\left(\frac{u_n}{u_{n+1}}-1\right)-1=\frac{bn-an}{a+n}-1=\frac{n(b-a)-a-n}{a+n}$$

$$=\frac{n-a-n}{a+n}, \text{ as } b-a=1$$

$$\text{or, } \left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n=\frac{-a}{a+n}\log n=\frac{-an}{a+n}\cdot\frac{\log n}{n}$$

$$=\frac{-a}{\frac{a}{n}+1}\cdot\frac{\log n}{n}$$

$$\text{or, } \lim_{n\to\infty}\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right]=-a\cdot 0=0<1.$$

$$\text{as } \lim_{n\to\infty}\frac{\log n}{n}=0.$$

Hence, the given series is divergent when $b-a=1$, by De-Morgan and Bertrand's test.

Q. 3. Test the convergence of the series, $(x>0)$

$$1+\frac{2^2}{3^2}x+\frac{2^2\cdot 4^2}{3^2\cdot 5^2}x^2+\frac{2^2\cdot 4^2\cdot 6^2}{3^2\cdot 5^2\cdot 7^2}x^3+\cdots \text{ to }\infty.$$

Soln: Let the general term be denoted by u_n.

$$\text{Then } u_n=\frac{2^2\cdot 4^2\cdot 6^2\cdots(2n)^2}{3^2\cdot 5^2\cdot 7^2\cdots(2n+1)^2}x^n$$

$$\text{and } u_{n+1}=\frac{2^2\cdot 4^2\cdot 6^2\cdots(2n)^2(2n+2)^2}{3^2\cdot 5^2\cdot 7^2\cdots(2n+1)^2(2n+3)^2}x^{n+1}$$

$$\therefore \frac{u_n}{u_{n+1}}=\frac{(2n+3)^2}{(2n+2)^2}\cdot\frac{1}{x}=\frac{\left(2+\frac{3}{n}\right)^2}{\left(2+\frac{2}{n}\right)^2}\cdot\frac{1}{x}$$

or, $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}$. Hence, D' Alembert's ratio test suggest's that the given series is convergent or divergent according as $\frac{1}{x} >$ or <1,

that is, $x <$ or >1.

If $x=1$, then D' Alembert's ratio test is unable to draw any conclusion.

Now, Raabe's test will be helpful.

We have, $\frac{u_n}{u_{n+1}} - 1 = \frac{(2n+3)^2}{(2n+2)^2} - 1 = \frac{\cancel{4n^2}+12n+9-\cancel{4n^2}-8n-4}{(2n+2)^2}$

$$= \frac{4n+5}{(2n+2)^2}$$

or, $n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{4n^2+5n}{(2n+2)^2} = \frac{4+\frac{5}{n}}{\left(2+\frac{2}{n}\right)^2}$

or, $\lim_{n\to\infty} \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right) \right\} = \frac{\cancel{4}}{\cancel{4}} = 1$

Hence, Raabe's test is helpless to infer any conclusion.

We have, $n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 = \frac{4n^2+5n}{(2n+2)^2} - 1 = \frac{\cancel{4n^2}+5n-\cancel{4n^2}-8n-4}{(2n+2)^2}$

or, $\left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 \right\} \log n = \frac{-3n-4}{(2n+2)^2} \cdot \log n = \frac{-3n^2-4n}{(2n+2)^2} \cdot \frac{\log n}{n}$

$$= \frac{3-\frac{4}{n}}{\left(2+\frac{2}{n}\right)^2} \cdot \frac{\log n}{n}$$

or, $\lim_{n\to\infty} \left[\left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 \right\} \right] = \frac{3}{4} \cdot 0 = 0 < 1$.

as $\lim_{n\to\infty} \frac{\log n}{n} = 0$

Hence, De Morgan and Bertrand's test infers that the given series is divergent when $x=1$.